Available online at www.sciencedirect.com

ScienceDirect

Physica D 220 (2006) 175–182

PHYSICA D

www.elsevier.com/locate/physd

On a recursive method for the estimation of unknown parameters of partially observed chaotic systems

Inés P. Mariño[a,*], Joaquín Míguez[b]

[a] *Nonlinear Dynamics and Chaos Group, Departamento de Matemáticas y Física Aplicadas y Ciencias de la Naturaleza, Universidad Rey Juan Carlos, C/ Tulipán s/n, 28933 Móstoles, Madrid, Spain*

[b] *Departamento de Teoría de la Señal y Comunicaciones, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés, Madrid, Spain*

Received 9 November 2005; received in revised form 17 April 2006; accepted 7 July 2006
Available online 4 August 2006
Communicated by R. Roy

Abstract

We investigate a recently proposed method for on-line parameter estimation and synchronization in chaotic systems. This novel technique has been shown effective to estimate a single unknown parameter of a primary chaotic system with known functional form that is only partially observed through a scalar time series. It works by periodically updating the parameter of interest in a secondary system, with the same functional form as the primary one *but no explicit coupling* between their dynamic variables, in order to minimize a suitably defined cost function. In this paper, we review the basics of the method, and investigate its robustness and new extensions. In particular, we study the performance of the novel technique in the presence of noise (either observational, i.e., an additive contamination of the observed time series, or dynamical, i.e., a random perturbation of the system dynamics) and when there is a mismatch between the primary and secondary systems. Numerical results, including comparisons with other techniques, are presented. Finally, we investigate the extension of the original method to perform the estimation of two unknown parameters and illustrate its effectiveness by means of computer simulations.

Keywords: Parameter estimation; Chaos synchronization; Chaos control

1. Introduction

Many problems in science and engineering reduce to adjusting the parameters of a dynamical model in order to match an observed time series. If we assume that (a) the model is a replica of the system originating the observations, except for the parameters to be adjusted, and (b) after a proper selection of its adjustable parameter values the model dynamics follows the observed time series closely, then the model parameters are *estimates* of the real system parameters [1].

The methods proposed in the literature to address this problem can be classified as either off-line or on-line techniques. Off-line methods first collect a set of observations (e.g., samples from the received time series) and then process the complete set iteratively to produce a sequence of approximations to the values of the unknown parameters. On-line techniques process the observations sequentially and parameter estimates are computed recursively, i.e., existing estimates are updated using only newly collected observations instead of the complete set of data. Off-line algorithms are computationally heavier, although often more accurate, and unsuitable for applications in which the model system should operate continuously.

Different procedures of both classes have been suggested. The so-called multiple shooting methods [2,3] are off-line techniques that address the estimation of the unknown fixed parameters, as well as the values of the dynamic variables of the chaotic system at a grid of sampling times, as a boundary-value problem and have shown to be very effective for several applications. However, they involve the optimization of large dimensional cost functions (not only the parameters are estimated) and can be complex to implement compared to other methods. Some standard

* Corresponding author. Tel.: +34 916647448; fax: +34 914887338.
E-mail address: ines.perez@urjc.es (I.P. Mariño).

0167-2789/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2006.07.008

statistical procedures for off-line estimation have also been proposed, including approximate maximum likelihood, least squares and moment-matching procedures [4,5]. Some authors have also extended classical Bayesian methods for sequential (on-line) inference, including the extended Kalman filter [6] and the unscented Kalman filter [7]. The latter has been shown effective in some scenarios but, in general, these techniques require several approximations, such as linearization and/or time discretization, as well as the assumption that any random perturbations (i.e., noise) be Gaussian with known mean and variance.

More recently, it has been shown that the synchronization properties of coupled chaotic systems can be advantageously exploited to design simpler parameter estimation algorithms. Assume the observed time series consists of a fraction of the dynamic state of a *primary* chaotic system with known functional form but some unknown parameters. The model, which we will hereafter refer to as the *secondary* system, is defined with the same equations, but the parameters that are unknown in the primary system are *adjustable* in the secondary one. Furthermore, the dynamic variables of both systems are coupled through the observed time series in a way that would ensure synchronization if their parameters were matched. With this setup, joint synchronization and parameter estimation can be attained [1,8–17]. This general approach followed the work of Parlitz [1], who proposed an adaptive control scheme driven by the synchronization error as a method for parameter estimation, and it is appealing because only the unknown parameters need to be estimated, which leads to lower dimensional problems. Both off-line [8,13,16,17] and on-line [1,9–12,14,15] techniques have been suggested. The latter are simpler and enable the continuous operation of the secondary system. The various techniques differ in the type of coupling between the systems and the way the synchronization error is defined. In all cases, the unknown parameters are handled as time-varying magnitudes and suitable differential equations are designed for them.

A more challenging problem arises when there is no explicit coupling between the dynamic variables of the chaotic systems, hence control over the secondary system must be exercised by the adaptation of its adjustable parameters alone. In [18] it was shown that if the time series consists of the full state of the primary system and there is a single parameter to be adjusted in the secondary system, identical synchronization and parameter estimation can be achieved (on-line) for certain types of systems. An alternative on-line method was very recently introduced in [19] that enables the estimation of a scalar parameter from the observation of a 1-dimensional time series, instead of the full primary-system state as in [18]. The key features of the technique in [19], compared to other synchronization-based on-line methods, are the following.

- It does not require an explicit coupling between the primary and secondary system, which makes this method depart clearly from all previous techniques except [18].
- It is aimed at the on-line minimization of cost functions that involve n-th order derivatives ($n > 1$) of the time series and the state variables of the secondary system. As a consequence, they become explicit functions of the adjustable parameters and can be handled analytically (with certain approximations).
- The on-line minimization is carried out by a discrete-time algorithm, while existing techniques handle the adjustable parameters as continuous-time variables that evolve according to adequately designed ordinary differential equations (ODEs) [1,9–12,14,15]. Although they can often be cast into ODEs by letting the time increments vanish, discrete-time procedures are practically better suited for implementation using digital computing devices. In the following, we refer to the method of [19] as discrete-time recursive update (DTRU).

In this paper, we review the basics of the DTRU method, investigate its robustness in the presence of noise and introduce some extensions. In particular, a formal problem statement and a general description of the DTRU procedure, which accounts for unknown parameter vectors with arbitrary dimension, is given in Section 2. In Section 3, we use the well-known Lorenz equations to numerically study the robustness of the method in the presence of noise and mismatches between the primary and secondary systems. By means of extensive computer simulations, we demonstrate the performance of the DTRU algorithm when: (a) there exist errors in the fixed (non-adjustable) parameters of the secondary system with respect to their counterparts in the primary one, (b) there is additive white Gaussian noise contaminating the observed time series and (c) the primary system is affected by dynamical noise which is not accounted for in the secondary one. In Section 4, we investigate the extension of the DTRU method to jointly estimate two parameters. Specifically, we show procedures for jointly estimating two parameters of both the Lorenz system (using a 1-dimensional time series as input) and a higher dimensional primary system built by diffusively coupling two Lorenz subsystems with different parameter sets. Finally, Section 5 is devoted to a brief discussion and concluding remarks.

2. Problem statement and proposed method

Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

represent the primary system with state variables $\mathbf{x} \in \mathbb{R}^n$, and unknown parameters $\mathbf{p} \in \mathbb{R}^m$. Given the functional form of Eq. (1), we can build the secondary system as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{q}) \tag{2}$$

where $\mathbf{y} \in \mathbb{R}^n$ are the state variables and $\mathbf{q} \in \mathbb{R}^m$ are the parameters to be adjusted. The system in Eq. (2) is fully observed, and we assume the ability to periodically update the value of $\mathbf{q}$. There is no coupling between the dynamical variables of the primary and secondary systems, but the observed time series $\mathbf{h}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^k$, which consists of a known transformation of a subset of the dynamical variables in Eq. (1), can be used for the update of $\mathbf{q}$. The goal is to derive an algorithm to adaptively adjust the secondary system parameters until the system variables, and the parameters themselves, converge to their counterparts in the primary system, i.e., both $\mathbf{y} \to \mathbf{x}$ and $\mathbf{q} \to \mathbf{p}$. In this way, identical synchronization between both systems is achieved and the unknown parameters of the primary system are estimated. With this aim, we revisit the estimation method proposed in [19].

Parameter estimation and identical synchronization can be jointly achieved through the minimization of a suitable cost function, denoted as $J(\mathbf{q}, t)$. We consider functions of the form

$$J(\mathbf{q}, t) = \int_0^t \|\mathbf{e}(\tau)\|^a \lambda^{\frac{(t-\tau)}{T}} \mathrm{d}\tau, \tag{3}$$

where $\mathbf{e}(\tau) = \mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y})$ is an error signal, $\|\cdot\|$ denotes vector norm, $a \geq 1$, T is the adaptation period (i.e., we assume that $\mathbf{q}$ can be updated every T time units) and $0 < \lambda < 1$ is a *forgetting factor* that emphasizes recent observations over older ones. Unless otherwise necessary to avoid ambiguities, we omit the time dependence of the dynamic variables ($\mathbf{x}$ and $\mathbf{y}$) all through the paper. We note that, when the systems attain identical synchronization ($\mathbf{y} \to \mathbf{x}$), the error signal vanishes ($\mathbf{e}(\tau) \to \mathbf{0}$) and, as a consequence, so does the cost function ($J(\mathbf{q}, nT) \to 0$). Since, in general, identical synchronization can only be achieved when $\mathbf{q} \to \mathbf{p}$, the minimizers in the sequence

$$\mathbf{q}_n = \arg\min_{\mathbf{q}} \{J(\mathbf{q}, nT)\}, \quad n \in \mathbb{N}, \tag{4}$$

are legitimate estimates of $\mathbf{p}$.

The practical applicability of this technique obviously relies on a choice of J that is tractable for effective on-line minimization. It will be shown, by way of the examples in Sections 3 and 4, that it is advantageous to define the error signal as a difference between (n-th order) derivatives of the dynamic variables, e.g., $\mathbf{e}(\tau) = \dot{\mathbf{x}} - \dot{\mathbf{y}}$ for a full-dimensional time series, instead of the straightforward difference between the state variables, $\mathbf{x} - \mathbf{y}$. The reason is that the time derivatives of the dynamic variables of the secondary system, $\dot{\mathbf{y}}$, are explicit functions of the parameters to be adjusted, and this fact can be exploited to analytically approximate the sequence of estimates in Eq. (4) using a simple recursive procedure. We refer to this general approach as discrete-time recursive update (DTRU) of the parameters.

A drawback of defining J in this way is that minimizing the mismatch between time derivatives of the state variables does not necessarily lead to identical synchronization and, although some preliminary results indicate that accurate parameter estimation can be achieved even without identical synchronization (i.e., by taking advantage of some form of generalized synchronization), further work is needed in this direction. Therefore, the value of the DTRU method should be found in its simplicity and its appeal to practical implementation using standard discrete-time (digital) devices.

3. Estimation of a single parameter in the presence of noise

In order to demonstrate the application of the DTRU method, we assume a Lorenz primary system,

$$\begin{aligned} \dot{x}_1 &= -\sigma_1(x_1 - x_2), \\ \dot{x}_2 &= R_1 x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= -b_1 x_3 + x_1 x_2, \end{aligned} \tag{5}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the system dynamic state and (σ_1, R_1, b_1) is the complete parameter vector. We use the same functional model for the secondary system, i.e.,

$$\begin{aligned} \dot{y}_1 &= -\sigma_2(y_1 - y_2), \\ \dot{y}_2 &= R_2 y_1 - y_2 - y_1 y_3, \\ \dot{y}_3 &= -b_2 y_3 + y_1 y_2, \end{aligned} \tag{6}$$

where $\mathbf{y} = (y_1, y_2, y_3)$ are the state variables and (σ_2, R_2, b_2) are the parameters.

3.1. Estimation of R_1

Let us consider first the problem of estimating R_1 when (σ_1, b_1) are known. In this case, we set $(\sigma_2, b_2) = (\sigma_1, b_1)$ from the start and the vectors of unknown and adjustable parameters, $\mathbf{p}$ and $\mathbf{q}$ in Eqs. (1) and (2), respectively, reduce to scalars, $p = R_1$ and $q = R_2$. In order to obtain estimates of the form (4), we wish to choose an error signal which is an explicit function of R_2. One such signal is $e(\tau) = \ddot{x}_1 - \ddot{y}_1$, hence we assume that the observed time series is $h(x_1) = \ddot{x}_1$ [1] and define the cost function

$$\begin{aligned} J(R_2, nT) &= \int_0^{nT} \lambda^{n-\frac{\tau}{T}} e^2(\tau) \mathrm{d}\tau \\ &= \int_0^{nT} \lambda^{n-\frac{\tau}{T}} (\ddot{x}_1 - \ddot{y}_1)^2 \mathrm{d}\tau, \end{aligned} \tag{7}$$

to be minimized with respect to R_2 (recall we omit the time dependence of the dynamic variables). In particular, the n-th update of the unknown parameter is carried out by solving

$$\frac{\mathrm{d}J}{\mathrm{d}R_2} = -2 \int_0^{nT} (\ddot{x}_1 - \ddot{y}_1) \frac{\mathrm{d}\ddot{y}_1}{\mathrm{d}R_2} \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau = 0 \tag{8}$$

for R_2. Clearly, there is a difficulty in the computation of the derivative $\mathrm{d}\ddot{y}_1/\mathrm{d}R_2$ because all y_i variables, $i = 1, 2, 3$, have an implicit dependence on R_2 (see Eq. (6)). However, the problem becomes very simple if we consider only the *explicit* derivatives, as proposed in [19]. Specifically, Eq. (8) reduces to

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}R_2} &\approx -2\sigma_2 \int_0^{nT} \lambda^{n-\frac{\tau}{T}} (\ddot{x}_1 - \ddot{y}_1) y_1 \mathrm{d}\tau \\ &= -2\sigma_2 \int_0^{nT} \lambda^{n-\frac{\tau}{T}} [\ddot{x}_1 - \sigma_2(R_2 y_1 + \sigma_2(y_1 - y_2) \\ &\quad - y_2 - y_1 y_3)] y_1 \mathrm{d}\tau, \end{aligned} \tag{9}$$

[1] We note that if x_i is the actual observable, it is straightforward to obtain $\dot{x}_i$ or $\ddot{x}_i$ with either simple circuitry or by means of numerical methods.

where the second line follows from explicitly writing $\ddot{y}_1$ in terms of $\mathbf{y}$ and σ_2. Solving for R_2 in Eq. (9) yields the n-th parameter estimate

$$R_{2,n} = \frac{\int_0^{nT} \lambda^{n-\frac{\tau}{T}}[\ddot{x}_1 - \sigma_2(\sigma_2(y_1 - y_2) - y_2 - y_1 y_3)]y_1 \mathrm{d}\tau}{\sigma_2 \int_0^{nT} \lambda^{n-\frac{\tau}{T}} y_1^2 \mathrm{d}\tau}, \tag{10}$$

which can be rewritten in the explicitly recursive form

$$R_{2,n} = R_{2,n-1} + \frac{A_n - R_{2,n-1} B_n}{\lambda C_{n-1} + B_n}, \tag{11}$$

with

$$A_n = \int_{(n-1)T}^{nT} [\ddot{x}_1 + \sigma_2(\dot{y}_1 + y_2 + y_1 y_3)]\sigma_2 y_1 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau, \tag{12}$$

$$B_n = \int_{(n-1)T}^{nT} (\sigma_2 y_1)^2 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau,$$

$$C_{n-1} = \int_0^{(n-1)T} (\sigma_2 y_1)^2 \lambda^{(n-1)-\frac{\tau}{T}} \mathrm{d}\tau, \tag{13}$$

by means of elementary algebraic manipulations. The latter formulas are particularly suitable for real-time application of the proposed technique.

3.2. Mismatched fixed parameters

The ability of the proposed method to estimate scalar parameters in an 'ideal' setup with perfect knowledge of the fixed parameters and noiseless systems and observations was already demonstrated in [19]. Here, we turn our attention to scenarios where these idealized assumptions do not hold.

As a first example, let us consider the situation in which the fixed parameters of the secondary system have a mismatch with respect to their counterparts in the primary system. In particular, we apply the recursive algorithm of Eq. (11) when $(\sigma_1, R_1, b_1) = (10, 28, 8/3)$ for the primary system (i.e., the standard parameter values) while $\sigma_2 = k\sigma_1$ and $b_2 = kb_1$, where $k = 1.03$ (i.e., there is a 3% error in both fixed parameters), in the secondary one. The two systems are numerically integrated using a second order Runge–Kutta (RK2) algorithm with time step $I = 5 \times 10^{-5}$ time units (t.u.) and the DTRU method is run with forgetting factor $\lambda = 0.995$ and update period $T = I = 5 \times 10^{-5}$ t.u. (for an easier approximation of the integrals in Eq. (11)). The initial conditions for the dynamic variables are chosen randomly, namely $y_1, x_1 \sim \mathcal{U}(12, 16)$, $y_2, x_2 \sim \mathcal{U}(-12, -16)$ and $y_3, x_3 \sim \mathcal{U}(19, 21)$, where $\mathcal{U}(a, b)$ denotes the uniform probability distribution in the interval (a, b). Note that, although identically distributed, the initial values of y_i and x_i are drawn independently. The starting value of R_2 is $R_{2,0} = R_1 - 10$.

Fig. 1 shows the absolute synchronization error, $|x_1 - y_1|$, which is achieved with this setup. It is observed that the two systems attain a relatively quick synchronization (a steady state error of $\approx$0.1 is reached after $\approx$10 t.u.), although there is an error floor that cannot be avoided because the primary and secondary systems are not identical.

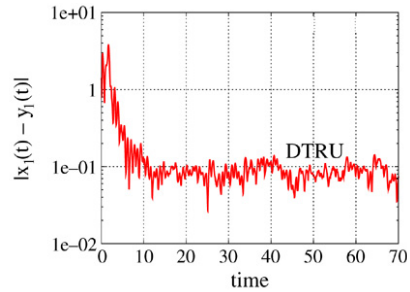

Fig. 1. Absolute synchronization error ($|x_1 - y_1|$) when applying the proposed DTRU algorithm of Eq. (11) to adjust parameter R_2 and there is a 3% mismatch between the parameter values (σ_1, b_1) of the primary system and their counterparts (σ_2, b_2) in the secondary one.

It is also interesting to study the effect of the mismatch on the estimation of the unknown parameter R_1. With that aim, we have carried out 30 independent computer simulations and computed the mean normalized absolute error (MNAE)

$$\mathrm{MNAE}_{30} = \frac{1}{30} \sum_{i=1}^{30} \left| \frac{R_2^{(i)} - R_1}{R_1} \right|, \tag{14}$$

where $R_2^{(i)}$ is the average value of the adjustable parameter R_2, after convergence of the DTRU algorithm, in the i-th simulation run.

The results are shown in Fig. 2, where $100 \times \mathrm{MNAE}_{30}$ is plotted versus the mismatch in the fixed parameters (σ_2, b_2). We observe, in Fig. 2(a), that the MNAE grows faster than the mismatch, until a 'saturation' value is reached ($\approx$66% of R_1). However, we have also verified that, for a relatively small mismatch of the fixed parameters ($\leq$5%), the resulting estimation error is approximately of the same order. This is shown in Fig. 2(b).

3.3. Dynamical Gaussian noise

In the derivation of Eq. (11) we have assumed that the observed time series is noiseless. However, this is not the usual situation in practice and we have carried out computer simulations to assess the robustness of the proposed method when there is Gaussian noise contaminating the observed signals. We first consider the case in which the dynamics of the primary system are perturbed by an additive noise term in one of the equations. Specifically, we assume that variable x_3 in the primary system evolves according to $\dot{x}_3 = -b_1 x_3 + x_1 x_2 + \xi$, where $\xi(t)$ is a continuous-time white Gaussian process with zero mean and power spectral density (PSD) $P_\xi = \frac{1}{2}$. The equations for $\dot{x}_1$ and $\dot{x}_2$ remain unchanged. The time series $\ddot{x}_1$ is generated using the RK2 algorithm with time step $I = 10^{-4}$ t.u. to integrate the (stochastic) primary system and we apply the DTRU method of Eq. (11) to estimate R_1 and synchronize the secondary system with the primary one. In this case, we assume perfect knowledge of the fixed parameters, i.e., $(\sigma_2, b_2) = (\sigma_1, b_1) = (10, \frac{8}{3})$.

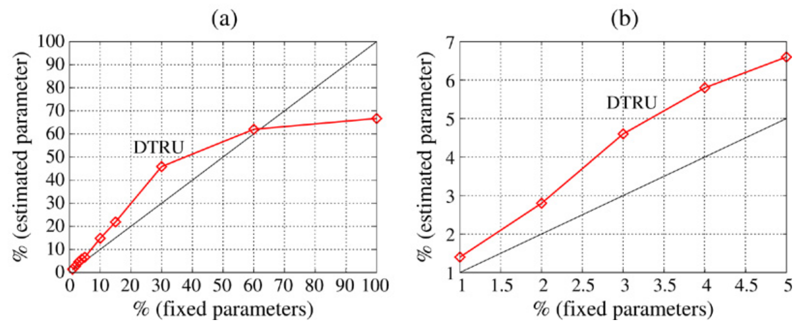

Fig. 2. MNAE in the estimation of the unknown parameter R_1 for several values of the fixed-parameter mismatch. The MNAE is rescaled between 0 and 100 for direct comparison with the mismatch. (a) The fixed-parameter mismatch varies between 0 and 100. (b) Zoom of plot (a), for values of the fixed-parameter mismatch between 1 and 5%.

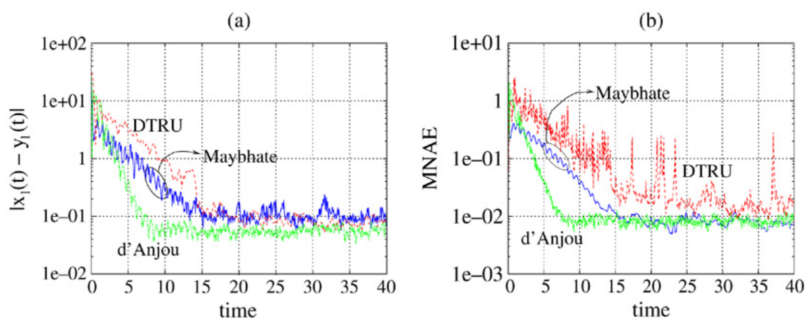

Fig. 3. Averaged numerical results when the primary system is perturbed with dynamical noise. (a) Mean absolute synchronization error ($|x_1 - y_1|$). (b) Mean normalized absolute error in the estimation of R_1.

In order to assess the performance of the DTRU technique, we have also applied the on-line parameter estimation methods proposed in [10] and [12] to this problem. The algorithm in [10], hereafter referred to as Maybhate's, relies on a linear feedback coupling of the Lorenz systems and the adjustable parameter R_2 is handled as an extra dynamic variable that evolves according to a suitably defined differential equation. We have integrated the equations with a feedback coefficient $\epsilon = 20$ and 'stiffness' constant $\delta = 5$, chosen to speed up the convergence of R_2. The technique in [12] is termed d'Anjou's in the following. In this case, the secondary system is driven by the primary one using the scheme of Pecora and Carroll [20] but the parameter R_2 is also handled as a continuous-time variable, whose trajectory is given by a certain differential equation. We set $\lambda = (\sigma_1+8)/4$ in [12, Eq. (12)] for fast convergence or R_2. Note that both techniques are heavily dependent on the synchronization properties of coupled chaotic systems, while the DTRU method does not require an explicit coupling between the dynamic variables of the primary and secondary systems.

Fig. 3(a) shows the absolute synchronization error, $|x_1 - y_1|$, attained by the three methods. The results are averaged over 30 independent simulation trials. In all simulations, the adjustable parameter R_2 is initialized with the value $R_2 = R_1 - 10$ for the three methods, while the dynamic variables are drawn randomly, namely $x_1, x_2 \sim \mathcal{U}(-16, -12)$, $y_1, y_2 \sim \mathcal{U}(12, 16)$, $x_3, y_3 \sim \mathcal{U}(19, 21)$. It is observed that the three techniques attain a very similar steady-state error, and the methods based on coupling are only slightly faster than the DTRU technique.

The MNAE_{30} in the estimation of R_1 for the same set of 30 simulations is shown in Fig. 3(b). The results are similar, although it is seen that the estimates provided by the DTRU method suffer from enhanced misadjustment noise, compared to Maybhate's and d'Anjou's methods.

3.4. Observational Gaussian noise

Another scenario of practical interest is one where the observations are contaminated with additive noise. Therefore, we have carried out additional numerical experiments in which the available times series (used to estimate R_1) is $\ddot{x}_1 + \xi$, where $\xi(t)$ is a zero-mean white Gaussian random process. The PSD of the stochastic process, P_ξ, is chosen so as to ensure that each discrete sample of the series in the simulation has variance $\sigma_\xi^2 = 5$. As in the previous experiments, we use the RK2 procedure to integrate both the primary and the secondary systems, with time step $I = 5 \times 10^{-4}$ t.u. The forgetting factor and update period of the DTRU algorithm are chosen as $\lambda = 0.97$ and $T = I = 5 \times 10^{-4}$ t.u., respectively. The values of the parameters in the primary system are standard.

For comparison, we have also applied Maybhate's and d'Anjou's techniques to the same problem. The specific parameters of these methods are given the same values as in Section 3.3. It is relevant to this experiment that Maybhate's procedure requires that the two Lorenz systems be coupled only through one equation (the differential equation for $\dot{y}_1$), while in d'Anjou's technique two equations of the secondary system are driven by the observed time series (namely, those for $\dot{y}_2$ and $\dot{y}_3$). Moreover, d'Anjou's method uses the series x_1 and $\dot{x}_1$

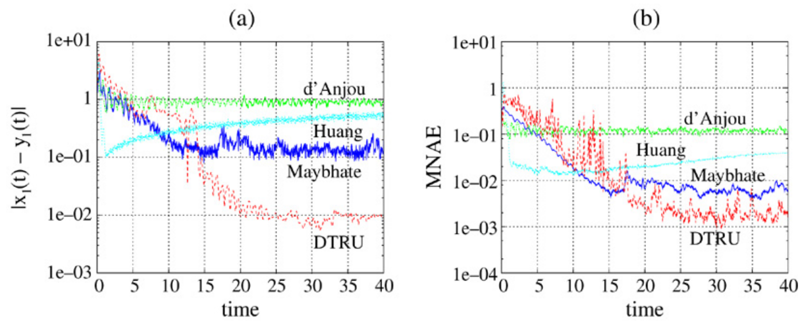

Fig. 4. Average performance results when the observed time series are contaminated with additive white Gaussian noise. (a) Mean absolute synchronization error ($|x_1 - y_1|$). (b) Mean normalized absolute error in the estimation of R_1.

simultaneously to estimate R_1, while Maybhate's needs only use x_1 (and only $\ddot{x}_1$ is needed for the DTRU procedure). All observed signals are assumed to be noisy.

For a further characterization of the effect of observational noise on the coupled system, we have also applied the method of Huang [15] to this problem. The distinct feature of the latter technique, compared to Maybhate's and d'Anjou's, is that it requires the observation of the full primary system state, **x**, and the linear feedback coupling of all equations in the secondary system. For this reason (the dynamics of the primary system are fully observed), Huang's method can attain much faster convergence than the other considered techniques, but can be subject to a stronger perturbation when the observations are noisy.

Fig. 4(a) shows the absolute synchronization error, $|x_1 - y_1|$, attained by the four methods and averaged over 100 independent simulations. For every run, the adjustable parameter R_2 is initialized with the value $R_2 = R_1 - 10$, while the dynamic variables are drawn randomly from uniform distributions, in particular $x_1, y_1 \sim \mathcal{U}(8, 16)$, $x_2, y_2 \sim \mathcal{U}(-16, -8)$, $x_3, y_3 \sim \mathcal{U}(18, 22)$. It is observed that the proposed DTRU method outperforms the other three techniques in terms of the steady-state error. This is because coupling introduces 'extra noise' in the secondary system (not only the differential equation of the adjustable parameter, but also every coupled equation becomes noisy). Clearly, it is the simple Maybhate's technique that provides the lowest error among the methods based on coupling.

Fig. 4(b) shows the MNAE_{100} for the same set of simulations. The results are coherent with those of Fig. 4(a), and we see how the proposed DTRU procedure attains the lowest estimation error.

4. Estimation of multiple parameters

4.1. Alternate estimation of two parameters in the Lorenz system

In this section, we demonstrate the application of the proposed method to estimate multiple parameters of the primary system. As a first example, we again assume the primary system of Eq. (5), where only b_1 is known, while R_1 and σ_1 need to be estimated. The observed time series enables the computation of $\dot{x}_1$ and $\ddot{x}_1$ (e.g., if it consists of x_1). In the secondary system, we set $b_2 = b_1$, while R_2 and σ_2 are adaptively adjusted until synchronization. We adopt an 'alternate estimation' approach in which one parameter (R_2) is updated for a period of time while the other one (σ_2) is kept fixed. After this period, the algorithm 'switches' and the second parameter (σ_2) is updated during another period while the first one (R_2) remains fixed. By iterating the estimation of both parameters, convergence to the desired values is attained and both chaotic systems synchronize. The n-th update of the adjustable parameter σ_2 is carried out by solving

$$\frac{\mathrm{d}J}{\mathrm{d}\sigma_2} = -\int_0^{nT} 2(\dot{x}_1 - \dot{y}_1)\frac{\mathrm{d}\dot{y}_1}{\mathrm{d}\sigma_2}\lambda^{n-\frac{\tau}{T}}\,\mathrm{d}\tau = 0 \tag{15}$$

for σ_2, which, using the same procedure as in Section 3, leads to

$$\sigma_{2,n} = -\frac{\int_0^{nT} \dot{x}_1(y_1 - y_2)\lambda^{n-\frac{\tau}{T}}\,\mathrm{d}\tau}{\int_0^{nT} (y_1 - y_2)^2\lambda^{n-\tau T}\,\mathrm{d}\tau}, \tag{16}$$

while the n-th estimate of R_2 is calculated according to Eq. (11). It is straightforward to put Eq. (16) in recursive form if needed.

We have carried out numerical simulations in which the primary Lorenz system is assigned the standard parameter values, and the algorithm starts with $\sigma_{2,n} = 15$, $R_{2,n} = 23$ and $\lambda = 0.94$. The equations of both systems have been numerically integrated using a fourth-order Runge–Kutta (RK4) method with integration step $I = 10^{-4}$ t.u. We have adopted the same value for the adaptation period of parameter R_2, $T_R = I = 10^{-4}$ t.u., while $T_\sigma = 10I = 10^{-3}$ t.u. is the update period for $\sigma_{2,n}$. The result of the proposed alternate estimation procedure is shown in Fig. 5. Convergence is slower than in the previous examples (because of the alternate procedure) and we have also verified that it can be more sensitive to the initial conditions of the algorithm and the choice of the update periods. Nevertheless, a considerable accuracy can also be achieved. We observe how both the absolute synchronization error, $|y_1 - x_1|$, and the normalized absolute error of both parameters, $|\frac{|\sigma_{2,n}-\sigma_1|}{\sigma_1}|$ and $|\frac{|R_{2,n}-R_1|}{R_1}|$, both fall below 10^{-9} after 2×10^4 t.u.

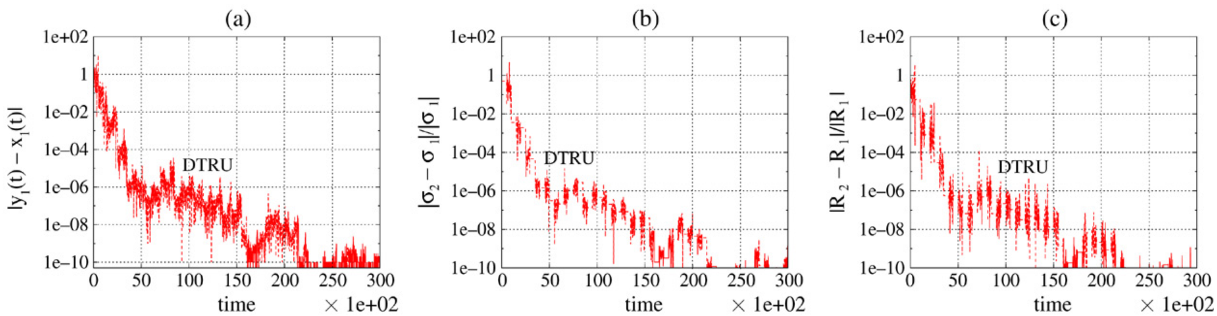

Fig. 5. Estimation of σ_1 and R_1 of the primary Lorenz system by adaptively adjusting σ_2 and R_2. (a) Temporal evolution of the absolute synchronization error, $|x_1 - y_1|$, as the parameters σ_2 and R_2 are adjusted. (b) and (c) Temporal evolution of the parameter normalized absolute errors, $|\frac{\sigma_{2,n}-\sigma_1}{\sigma_1}|$ and $|\frac{R_{2,n}-R_1}{R_1}|$, as the parameters σ_2 and R_2 are adjusted.

4.2. Joint estimation of two parameters in a 6-dimensional system

For the second example, we consider a 6-dimensional chaotic system that we build by diffusively coupling two Lorenz oscillators with different parameter sets. Specifically, the primary system is given by

$$\begin{aligned}
\dot{x}_1 &= -\sigma_1(x_1 - x_2) + \epsilon(x_4 - x_1),\\
\dot{x}_2 &= R_1 x_1 - x_2 - x_1 x_3,\\
\dot{x}_3 &= -b_1 x_3 + x_1 x_2,\\
\dot{x}_4 &= -\sigma_3(x_4 - x_5) + \epsilon(x_1 - x_4),\\
\dot{x}_5 &= R_3 x_4 - x_5 - x_4 x_6,\\
\dot{x}_6 &= -b_3 x_6 + x_4 x_5,
\end{aligned} \tag{17}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ are the dynamic variables, $(\sigma_1, R_1, b_1, \sigma_3, R_3, b_3)$ is the complete parameter vector and ϵ is the diffusion coefficient that determines the strength of the coupling between the two oscillators. The secondary system is

$$\begin{aligned}
\dot{y}_1 &= -\sigma_2(y_1 - y_2) + \epsilon(y_4 - y_1),\\
\dot{y}_2 &= R_2 y_1 - y_2 - y_1 y_3,\\
\dot{y}_3 &= -b_2 y_3 + y_1 y_2,\\
\dot{y}_4 &= -\sigma_4(y_4 - y_5) + \epsilon(y_1 - y_4),\\
\dot{y}_5 &= R_4 y_4 - y_5 - y_4 y_6,\\
\dot{y}_6 &= -b_4 y_6 + y_4 y_5,
\end{aligned} \tag{18}$$

where $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$ are the state variables and $(\sigma_2, R_2, b_2, \sigma_4, R_4, b_4)$ are the parameters. It should be noted that both the primary and secondary systems are true 6-dimensional oscillators, since (x_1, x_2, x_3) and (x_4, x_5, x_6) (correspondingly, (y_1, y_2, y_3) and (y_4, y_5, y_6)) do not synchronize because of their different parameter sets.

We consider the problem of estimating the parameter set $\mathbf{p} = (R_1, R_3) = (28, 32)$ when the derivative signals $(\ddot{x}_1, \ddot{x}_4)$ can be computed from the observed time series. The remaining primary parameters are assumed known, with values $(\sigma_1, b_1, \sigma_3, b_3) = (10, \frac{8}{3}, 18, 2)$, hence they can be fixed in the secondary system, i.e., $(\sigma_2, b_2, \sigma_4, b_4) = (\sigma_1, b_1, \sigma_3, b_3)$. The forgetting factor is set to $\lambda = 0.99$ and the diffusion coefficient is chosen as $\epsilon = 10$. We have numerically integrated the two systems using the RK4 method with time step $I = 10^{-4}$ t.u.

Instead of the alternate estimation scheme of the previous section, we design an algorithm which simultaneously updates the two adjustable parameters every $T = I = 10^{-4}$ t.u. The error signal from which the cost function is defined is $e(\tau) = \sum_{i\in\{1,4\}} \ddot{x}_i - \ddot{y}_i$. The usual analytic approximation by explicit derivatives yields the following updating rule for R_2,

$$R_{2,n} = \frac{\int_0^{nT} [\ddot{x}_1 + \sigma_2(\dot{y}_1 + y_2 + y_1 y_3) - \epsilon(\dot{y}_4 - \dot{y}_1)] y_1 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau}{\sigma_2 \int_0^{nT} y_1^2 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau}, \tag{19}$$

which can be easily put in recursive form. The initial value for the algorithm is $R_{2,0} = 33$. The algorithm for the estimation of R_3 that results from the error signal $e(\tau) = \sum_{i=1,4} \ddot{x}_i - \ddot{y}_i$ is similar to Eq. (19), but using the variables of the second subsystem instead, i.e.,

$$R_{4,n} = \frac{\int_0^{nT} [\ddot{x}_4 + \sigma_4(\dot{y}_4 + y_5 + y_4 y_6) - \epsilon(\dot{y}_1 - \dot{y}_4)] y_4 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau}{\sigma_2 \int_0^{nT} y_4^2 \lambda^{n-\frac{\tau}{T}} \mathrm{d}\tau}. \tag{20}$$

The algorithm starts with $R_{4,0} = 27$. We remark that the adaptation of R_2 and R_4 is carried out simultaneously.

The obtained numerical results are depicted in Fig. 6. Plots (a) and (b) show the evolution of the absolute synchronization errors $|y_1 - x_1|$ and $|y_4 - x_4|$, respectively. It is seen that both subsystems of the secondary oscillator synchronize with their counterparts in the primary oscillator. Thus, identical synchronization of the full 6-dimensional systems is achieved. As was the case in the previous experiments, the decay of the synchronization error is in pace with the parameter estimation error, as can be observed in plots (c) and (d), that present the normalized absolute deviations $|\frac{R_{2,n}-R_1}{R_1}|$ and $|\frac{R_{4,n}-R_3}{R_3}|$, respectively. Very accurate parameter estimation is achieved (both error curves fall below 10^{-9} after less than 80 t.u.).

5. Conclusions

We have investigated the recently proposed method of [19] for on-line parameter estimation and synchronization of chaotic systems without explicit coupling of the dynamic variables. In the latter work, only the problem of estimating a single unknown parameter was addressed. In this paper, we have

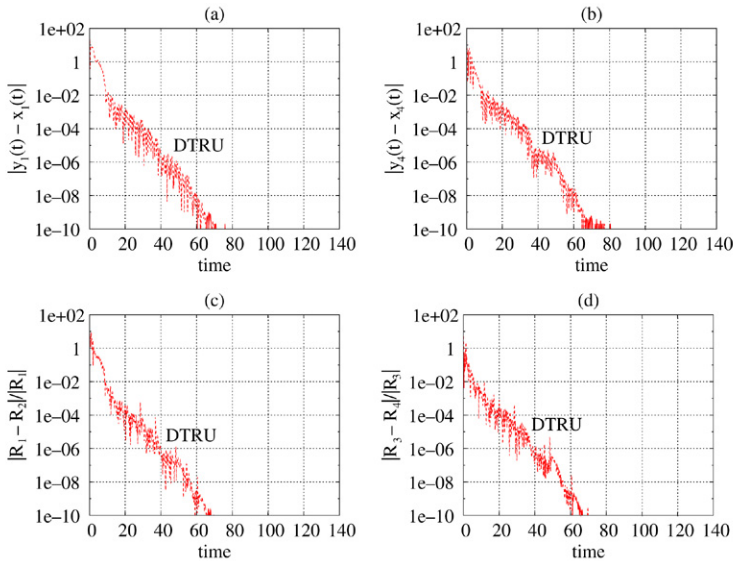

Fig. 6. Estimation of R_1 and R_3 of the primary 6-dimensional system by adaptively adjusting R_2 and R_4. (a) and (b) Temporal evolution of the absolute error between the first and fourth variables, respectively, of the primary and secondary systems, $|x_1 - y_1|$ and $|x_4 - y_4|$, as the parameters R_2 and R_4 are adjusted. (c) and (d) Temporal evolution of the normalized absolute deviations $|\frac{R_{2,n}-R_1}{R_1}|$ and $|\frac{R_{4,n}-R_3}{R_3}|$, as the parameters R_2 and R_4 are adjusted.

extended the basic statement of the method (to account for parameter vectors with arbitrary dimension) and shown, by means of numerical results, how it can be successfully applied in different scenarios. In particular, we have studied the estimation of a scalar parameter in noisy systems and the joint estimation of two parameters. For the first problem, we have considered different setups, including (a) mismatched fixed parameters in the secondary system, (b) a random dynamical perturbation of the primary system and (c) observational additive noise contaminating the available time series. The obtained results show that the performance of the proposed algorithm degrades smoothly because of mismatches in the fixed parameters, while the method is robust to both dynamical and observational noise. Specifically, our results indicate that its performance is similar to existing on-line parameter estimation techniques based on coupling when dynamical noise is present, and clearly superior when there exists observational noise. An interpretation of these results is that coupling the chaotic systems, while enhancing the accuracy and convergence speed of the estimation and synchronization methods in an ideal (noise free) setup, brings unexpected random perturbations into the dynamics of the secondary system that can flaw the parameter estimates and increase the synchronization error. Since our method does not require coupling the dynamic variables, it can be more robust to this type of noise. For the problem of multiple parameter estimation, we have numerically demonstrated that the proposed technique can be successfully applied to jointly estimate two parameters both in the Lorenz system and in a higher dimensional system formed by diffusively coupling two non-identical Lorenz oscillators.

Acknowledgements

Inés P. Mariño acknowledges the support of the Spanish Ministry of Education and Science of Spain (project BFM2003-03081). Joaquín Míguez acknowledges the support of Xunta de Galicia (project PGIDIT04TIC105008PR), the Ministry of Education and Science of Spain (project DOIRAS, ref. TIC2003-02602) and Comunidad de Madrid (project PROMULTIDIS-CM, ref. S0505/TIC/0223).

References

[1] U. Parlitz, Phys. Rev. Lett. 76 (1996) 1232.
[2] E. Baake, M. Baake, H.G. Bock, K.M. Briggs, Phys. Rev. A 45 (1992) 5524.
[3] A. Ghosh, V.R. Kumar, B.D. Kulkarni, Phys. Rev. E 64 (2001) 056222.
[4] J. Timmer, Chaos Solitons Fractals 11 (2000) 2571.
[5] V.F. Pisarenko, D. Sornette, Phys. Rev. E 69 (2004) 036122.
[6] H. Singer, J. Comput. Graph. Statist. 11 (2002) 972.
[7] A. Sitz, U. Schwarz, J. Kurths, H.U. Voss, Phys Rev. E 66 (2002) 016210.
[8] U. Parlitz, L. Junge, L. Kocarev, Phys. Rev. E 54 (1996) 6253.
[9] C. Zhou, C.-H. Lai, Phys. Rev. E 59 (1999) 6629.
[10] A. Maybhate, R.E. Amritkar, Phys. Rev. E 59 (1999) 284.
[11] A. Maybhate, R.E. Amritkar, Phys. Rev. E 61 (2000) 6461.
[12] A. d'Anjou, C. Sarasola, F.J. Torrealdea, R. Orduna, M. Graña, Phys. Rev. E 63 (2001) 046213.
[13] H. Sakaguchi, Phys. Rev. E 65 (2002) 027201.
[14] R. Konnur, Phys. Rev. E 67 (2003) 027204.
[15] D. Huang, Phys. Rev. E 69 (2004) 067201.
[16] C. Tao, Y. Zhang, G. Du, J.J. Jiang, Phys. Rev. E 69 (2004) 036204.
[17] I. Grosu, Int. J. Bifurcation Chaos 14 (2004) 2133.
[18] U.S. Freitas, E.E.N. Macau, C. Grebogi, Phys. Rev. E 71 (2005) 047203.
[19] I.P. Mariño, J. Míguez, Phys. Rev. E 72 (2005) 057202.
[20] L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. 64 (1990) 821.